Indian Statistical Institute

Final Examination 2015-2016

M.Math First Year

Functional Analysis

Time : 3 Hours Date : 29.04.2016 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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Note: (i) X, Y: Banach spaces. (ii) $\mathcal{H}$: Hilbert space. (iii) $\mathcal{B}(\cdots)$: set of bounded linear operators.

Q1. (10 marks) Given $\varphi \in (l^2)^*$, find $y \in l^2$ such that

$$\varphi(x) = \langle x, y \rangle_{l^2} \qquad (\forall x \in l^2).$$

Q2. (15 marks) Let $T \in \mathcal{B}(X)$. Prove that the approximate point spectrum of T is a compact set.

Q3. (10 marks) Let $\{x_n\} \subseteq X$ and $x_n \to x$ weakly. Prove that there exists a sequence $\{y_n\} \subseteq \text{span}\{x_n\}$ such that $y_n \to x$ in norm.

Q4. (15 marks) Prove that X is reflexive if and only if X^* is reflexive.

Q5. (10 marks) Let $T : X \to Y$ be a weakly continuous linear map. Prove that T is continuous.

Q6. (15 marks) Let X, Y be infinite dimensional spaces and $T \in \mathcal{B}(X, Y)$ a compact operator. Prove that $\text{ran}T$ can not contain an infinite dimensional closed subspace.

Q7. (15 marks) Let $A, B \in \mathcal{B}(\mathcal{H})$. Suppose that A is a positive operator and $AB = BA$. Prove that $\sqrt{A}B = B\sqrt{A}$.

Q8. (10 marks) Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $\varphi \in \mathcal{S}^*$. Prove that there exists $\tilde{\varphi} \in \mathcal{H}^*$ such that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$. Moreover, prove that $\tilde{\varphi}$ is unique.

Q9. (15 marks) Let $P_1, P_2 \in \mathcal{B}(\mathcal{H})$ be orthogonal projections. Prove that $P_1 + P_2$ is an orthogonal projection if and only if $P_1P_2 = 0$.